Lorentz transformations & 4-vectors

Now that we know that Einstein's relativity is invariant of
\[ S_{12}^2 = c^2 (t_2 - t_1)^2 - (\vec{r}_2 - \vec{r}_1)^2 \]
\[ = c^2 (t'_2 - t'_1)^2 - (\vec{r}'_2 - \vec{r}'_1)^2 \]
in any inertial frame.

We need to bite the bullet and find the transformations ("Lorentz")
of \( \vec{r}, t \rightarrow \vec{r}', t' \) of the coordinates of any spacetime point between \( O \) & \( O' \).
These are the generalization of
\[ t' = t \]
\[ \vec{r}' = \vec{r} - \vec{v} \cdot t \], Galileo's transforms, and should reduce to them in the "\( c \rightarrow \infty \) limit" (or \( \vec{v} \cdot \vec{r} / c \rightarrow 0 \) limit).

We'll argue via the close analogy of \( s^2 \) to distance \( (\vec{r}_1 - \vec{r}_2)^2 \) in ordinary Euclidean space. In \((x, y, z) \)-space \((\mathbb{R}^3)\) distances to the origin (or between any two points) are preserv-
ved by rotations in the $x$-$y$, $y$-$z$, $z$-$x$ plane, i.e.

\[ x^2 + y^2 + z^2 = \text{constant} \]

\[ \text{N.B.: every rotation preserving } S^2 \text{ can be decomposed into three rotations...} \]

\[ \text{SO}(3)'s \text{ "group of transformations"} \]

\[ \text{spheres in } \mathbb{R}^3 \text{ are invariant under rotations} \]

Similarly in 4d space,

\[ (t_1, \vec{r}_1) \rightarrow (t, \vec{r}) \quad \text{and} \quad \vec{r}_1 \rightarrow \vec{0} \]

"rotations" preserving $S^2 = c^2 t^2 - \vec{r}^2$ should consist of:

1. **Ordinary rotations**

   \[ \text{plane } x-y, \ y-z, \ z-x \]

   which preserve $S^2$ since clearly they preserve $t$ and $\vec{r}^2$.

2. **"Rotations"** in $t-x$, $t-y$, $t-z$ planes

   (these are called "hyperbolic" rotations, we'll see why)
recall usual notus, say in \( x-y \) plane:

\[
\begin{align*}
  x' &= z \\
  y' &= \cos \alpha x + \sin \alpha y \\
  y' &= -\sin \alpha x + \cos \alpha y
\end{align*}
\]

\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} =
\begin{pmatrix}
  \cos \alpha & \sin \alpha & 0 \\
  -\sin \alpha & \cos \alpha & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]

---

In the \( x-t \) plane, we similarly want to 'rotate \( t \)', but we want \( c^2 t^2 - x^2 = c^2 t'^2 - x'^2 \).

"Trick": let \( t = i T \), \( t' = i T' \) ("analytically continue")

\[
c^2 t^2 - x^2 = c^2 t'^2 - x'^2
\]

\[
-\tau^2 - x^2 = -\tau'^2 - x'^2
\]

\[
\tau^2 + x^2 = \tau'^2 + x'^2
\]

But now we know what the transform is, since this allowed us to reduce \( s^2 \) to usual Euclidean distance.

Transformation of \( \tau x \rightarrow \tau' x' \)

must be of the form \( x' = \cos \beta x + \sin \beta \tau \)

\[
\tau' = -\sin \beta x + \cos \beta \tau
\]

---

Take this as a trick to motivate eqns on bottom of p. 30, or top of p. 31.
But now, we know $\tau = -i \, c \, t$

$\tau' = -i \, c \, t$

So we have

\[
\begin{cases}
X' = \cos \psi \, x + \sin \psi \, (-i \, c \, t) \\
-\, i \, c \, t' = -\sin \psi \, x + \cos \psi \, (-i \, c \, t)
\end{cases}
\]

This is nonsensical, for now, since mixed real & imaginary coordinates, but if we take $\psi = -i \, \psi$

\[
\begin{align*}
\sin \psi &= \frac{e^{i \psi} - e^{-i \psi}}{2i} = \frac{-\psi - e^{+\psi}}{2i} \\
\cos \psi &= \frac{e^{i \psi} + e^{-i \psi}}{2} = \frac{-\psi + e^{+\psi}}{2}
\end{align*}
\]

we'll obtain from (x):

\[
x' = \cosh \psi \, x + i \, \sinh \psi \, (-i \, c \, t),
\]

so:

\[
\begin{align*}
X' &= \cosh \psi \, x + \sinh \psi \, c \, t, \\
\text{for } t': & \quad -\, i \, c \, t' = -\, i \, \sinh \psi \, x + \cosh \psi \, (-i \, c \, t) \\
& \quad = (-i) \, \sinh \psi \, x + (-i) \, \cosh \psi \, c \, t, \text{ equal } (-\, i),
\end{align*}
\]

so:

\[
c t' = \sinh \psi \, x + \cosh \psi \, c \, t
\]
Hence the transforms that preserve
\[ c^2 t^2 - x^2 = c^2 t'^2 - x'^2 \]
have the form
\[
\begin{align*}
x' &= \cosh \psi x + \sinh \psi ct \\
c't' &= \sinh \psi x + \cosh \psi ct
\end{align*}
\]
This is all good, but what is \( \psi \)?

Clearly, \( \psi \) must be related to
motion of \( O' \) wrt \( O \) with
some velocity \( V \) in the \( x \)
direction. The origin \( O \)
has coordinates \((t, \bar{x})\) in the
\( O \) frame. In the \( O' \) frame the
coordinates of the origin of \( O \) are
\[
\begin{align*}
x' &= \sinh \psi ct \\
c't' &= \cosh \psi ct
\end{align*}
\]
hence \( \frac{x'}{ct'} = \tanh \psi \)
This is velocity \( \frac{V}{c} \)
\( O \) wrt \( O' \) divided by \( c \)

\( \text{Note: for those who are reading G&L, note that} \)
\( \text{I am considering the same transform to} \)
\( \text{that in the book --- except I have the} \)
\( O \) \& \( O' \) \( \text{frames switched} \)
So, we have \[ \tanh \psi = \frac{V_x \text{ (of } \theta \text{ wrt } \theta')}{c} \] (or \[ = -\frac{V_x \text{ (of } \theta' \text{ wrt } \theta)}{c} \] )

Then, since \[ \cosh \psi = \frac{1}{\sqrt{1 - \tanh^2 \psi}} \]
\[ = \frac{\tanh \psi}{\sqrt{1 - \tanh^2 \psi}} \]
we have
\[ \cosh \psi = \frac{1}{\sqrt{1 - \frac{V_x^2}{c^2}}} \]
\[ \sinh \psi = \frac{V_x}{c \sqrt{1 - \frac{V_x^2}{c^2}}} \]

Proof. We basic
\[ \sinh \psi = \frac{e^\psi - e^{-\psi}}{2} \]
\[ \cosh \psi = \frac{e^\psi + e^{-\psi}}{2} \]
\[ \tanh \psi = \frac{e^\psi - e^{-\psi}}{e^\psi + e^{-\psi}} \]
\[ 1 - \tanh^2 \psi = \frac{1}{\cosh^2 \psi} \]
\[ \sinh^2 \psi = \frac{1 - \cosh^2 \psi}{2} \]
\[ \cos^2 \psi + \sinh^2 \psi = 1 \]
\[ \cos^2 \psi + \frac{V_x^2}{c^2} = 1 \]

\[ t' = \frac{V_x}{c} t + x \]

\[ x' = \frac{1}{\sqrt{1 - \frac{V_x^2}{c^2}}} x + \frac{V_x}{c} t \]
\[ \frac{V_x}{c} t = \frac{V_x}{c} t + \frac{1}{\sqrt{1 - \frac{V_x^2}{c^2}}} \]

\[ x' = x + \frac{1}{\sqrt{1 - \frac{V_x^2}{c^2}}} \]

\[ t' = \frac{V_x}{c} t \]

where \( V_x \) is the velocity of \( \theta \) wrt \( \theta' \)

(unprimed) (primed)
Go back to

\[ x' = y(x - \frac{v_x}{c} ct) \]
\[ ct' = y(ct - \frac{v_x}{c} x) \]
\[ y = \frac{1}{\sqrt{1 - v^2/c^2}} \in \mathbb{R}^2 \]

Points where \( x' = 0 \) will give the \( ct' \) axis in \( x-tc \) plane.

\( x' = 0 \implies x = \frac{v_x}{c} ct \) ; Note: \( ct' \) axis = worldline of \( \theta' \) (waves w/ \( v_x \)).

Points where \( t' = 0 \) will give \( x' \) axis in \( x-tc \) plane.

\( x' = 0 \implies ct = \frac{v_x}{c} x \)

Clearly, lines of constant \( x' \) & \( ct' \) are parallel to the \( ct' \) axis (constant \( x' \)) and the \( x' \) axis (constant \( t' \)), respectively.

Also clear that \( x' = ct' \) is the same lightcone.

"Hyperbolic" notation maps \( ct' \) & \( x' \) = constant lines to tilted lines on \((x,ct)\) diagram; the angle is easy to remember by (1) recall \( ct' \) axis = \( \theta' \) worldline (waves w/ \( v_x \) in \( x-ct \))
(2) lightcone is invariant, so \( x' \) axis symmetric to \( ct' \) are w/ lightcone.
An example of use:

let a rod be @ $x_A$ in $\Theta$

\[ x'_B - x'_A = \gamma (x_B - x_A) = \gamma \left( \frac{c t_B}{c} - \frac{c t_A}{c} \right) \]

\[ = \gamma \left( L - \frac{V}{c} (L + \tan \alpha) \right) = \gamma L \left( 1 - \tan^2 \alpha \right) \]

\[ = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} L \left( 1 - \frac{V^2}{c^2} \right) = L \sqrt{1 - \frac{V^2}{c^2}} \]

Moral: coordinates in tilted axes are not straightforward to read off diagram — but it's useful to see which events are simultaneous, or occur at same place, in which frame. (Distances in tilted $x',ct'$ not contraction)
Another ex. - of use of diagrams & causality --

- Suppose tachyons - moving faster than light - exist and for simplicity, allow for instantaneous transmission of information.

\[ A \text{ & } B \text{ two observers} \]

\[ A \text{ sends instantaneous info to } B \]

\[ C \text{ happens to be at same space point & same time as when } B \text{ gets signal.} \]

\[ \text{She can communicate immediately (as per her rest frame) with } D \]

\[ D \text{ happen to be at position of } A \text{ when he gets signal.} \]

\[ \text{But this arrives before } A \text{ even sent the signal to } B \text{ in the 1st place!} \]

Einstein

(ii) Relativity & instantaneous propagation of signals are incompatible.

(Also true for faster than light --)

(iii) For A & B signal 2 travels \[ \text{see H.W.} \]

(backward in time, for \[ C \text{ & } D \text{, signal 1} \]

i.e. timelike causally connected)

(iv) Events which are in each other's lightcones always have well defined time ordering, so no problem of causality...
These (bottom of p. 32) are the Lorentz
transform. There are, in fact, three more
(so-called "boosts") — in the y-t and z-t
plane. They look just like the above, but
\( x \to y \),
\( v_x \to v_y \) (and primes).

The sign in front of terms linear w.r.t. \( \frac{v_x}{c} \)
is best recalled by forgetting \( \sqrt{1 - \frac{v_x^2}{c^2}} \)
and noticing that
\[
x' = x + \frac{v_x}{c} t + O\left( \frac{v_x^2}{c^2} \right)
\]
\[
t' = t + O\left( \frac{v_x}{c} \right)
\]
So one gets Galilean transforms and these
tell us, clearly, that it is \( O' \) moving in
+ x direction w/ \( |v_x| \) (or \( O' \) moving \( m - x \) w/ \( v_x \)),
(because \( x' \) is increasing w/ \( t \) for fixed \( x \))
\[
0' \quad \begin{array}{c}
O' \quad \frac{\partial}{\partial t} \to v_x \\
\end{array}
\]
Summary: Interval \( \Delta s^2 = c^2 \Delta t^2 - \Delta x^2 \) is
preserved by \( 3 + 3 = 6 \) kind of "rotation"
- 3 rotations of space \((x,y,z)\)
- 3 "boosts" ("hyperbolic" rotations)
  in three directions \(\to (x', y', z')\)

...total of 6 parameters account for all the "proper transformations" (be added: P&T)
notations of space = \(SO(3)\) "group of transformations"

\[SO(3) = 3 \times 3\]
set of transforms of orthogonal matrices with unit determinant preserving \(\mathbb{R}^3\)

\[\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \hat{O} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \hat{O}^T \hat{O}^T\]

\(\hat{O} \in SO(3)\), acting on 3 vectors

\[\hat{O}^T \hat{O} = I \iff \text{orthogonal} \quad \text{Recall}\]

\[\begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix} \hat{O} \]

\[\begin{pmatrix} x_2' & y_2' & z_2' \end{pmatrix} = \begin{pmatrix} x_1' & y_1' & z_1' \end{pmatrix} \quad \text{rotation preserves} \quad \vec{r}_2' + \vec{r}_1' \]
For Lorentz transforms, this generalizes very nicely -- except (1) Lorentz transforms should involve not $3 \times 3$ but $4 \times 4$ matrices -- since they act not in $(x, y, z)$ but in $(ct, x, y, z)$ space.

(2) Lorentz transforms preserve not the $r_1 \cdot r_2$ inner product, but a more complicated one: $c^2 t_2 t_1 - r_1 \cdot r_2$

---

**NB:** don't get confused by this last statement... in analogy w/ rotations which preserve

$$(r_1 - r_2)^2 = (r_1 - r_2) \cdot (r_1 - r_2)$$

for all $r_1 \neq r_2$

we note that

$$(r_1 - r_2)^2 = r_1 \cdot r_1 - 2 r_1 \cdot r_2 - r_2 \cdot r_2$$

but this is $$(r_1 - \vec{d})^2$$

& must be invariant

Hence $$(r_2 - \vec{d})^2$$

& must be invariant

hence $r_2 \cdot r_1$ must be rotationally invariant (and it is)

> some logic for Lorentz: if $c^2(t_2 t_1) - (r_1 - r_2)^2$ is
invariant, for all \((t_1, \vec{r}_1) \neq (t_2, \vec{r}_2)\),
then it must be that \(c^2 t_2 t_1 - \vec{r}_1 \cdot \vec{r}_2\)
is also invariant.

Just like \(\vec{r}_1^2\), \(\vec{r}_1 \cdot \vec{r}_2\) notation is very
useful when considering non-relativistic physics
(where space is isotropic so laws are invariant
under rotations), it is useful to
introduce similar notation for \((ct, x, y, z)\).

Introduce

\[
\begin{align*}
X^0 &= ct \\
X^1 &= x \\
X^2 &= y \\
X^3 &= z
\end{align*}
\]

\[
(X^0, X^1, X^2, X^3) \equiv \{X^i, \ i = 0, 1, 2, 3\}
\]

\[
\{X^i\} = (ct, \vec{r})
\]

If we have another 4-vector, \(\{X^i\} = (ct, \vec{r})\),
we just argued that
\(c^2 t^2 - \vec{r} \cdot \vec{r}\) is invariant under Lorentz
transformations. Using \(X^i\), \(x^i\) notation
we write this in the following form: \(\to\)
\[
\sum_{i,j=0,1,2,3} \tilde{x}^i g_{ij} x^j = c^2 \tilde{t}\tilde{t} - \tilde{r}\tilde{r}
\]

where \( \| g_{ij} \| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = G \)

is called the "Minkowski space metric tensor".

This is a 2-index one 4×4 matrix

\[
\tilde{x}^0 g_{00} \tilde{x}^0 + \tilde{x}^1 g_{11} \tilde{x}^1 + \tilde{x}^2 g_{22} \tilde{x}^2 + \tilde{x}^3 g_{33} \tilde{x}^3 + 1 - 1 - 1 - 1
\]

since all other components of \( \| g_{ij} \| \) vanish.

\[
= c\tilde{t}\tilde{c} - \tilde{r}\tilde{r} - \tilde{g}_x - \tilde{g}_y - \tilde{g}_z = c^2 \tilde{t}\tilde{t} - \tilde{r}\tilde{r}
\]

(as promised)

In matrix notation:

\[
\sum_{ij} \tilde{x}^i g_{ij} x^j = X^T G \cdot X
\]

(Einstein's summation rule)

(a) \( i, j = 0, 1, 2, 3 \) understood

(b) omit summation signs: when indices repeat, \( \Sigma \) understood
where \( \tilde{X}^T = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \)

\[
G = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

\[
\tilde{X} = \begin{pmatrix}
\tilde{x}_0 \\
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\end{pmatrix}
\]

So we have \( \tilde{X}^T \cdot G \cdot \tilde{X} = c^2 \tilde{t}^2 - \tilde{r} \cdot \tilde{r} \)

(this is Lorentz invariant)

so is \( \tilde{r} \).

On \( \tilde{X} \) (or \( \tilde{X}^2 \), or any 4-vector) the Lorentz transforms act as a \( 4 \times 4 \) matrix

\[
X = \hat{O} \tilde{X}
\]

This represents \( 4 \times 4 \) Lorentz transform, e.g., \( \hat{O} = \begin{pmatrix}
1 & -\frac{\gamma c}{1+\frac{v^2}{c^2}} & 0 & 0 \\
\frac{\gamma v}{1+\frac{v^2}{c^2}} & \frac{v^2}{1+\frac{v^2}{c^2}} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\)

But \( \hat{O} \) can be more general - (rots & other boosts can be involved)
Similarly \( \tilde{X} = \Theta \tilde{X}' \).

\[ X = \Theta X' \]

 Relevant Lorentz transforms:

\[ \text{4 vector coordinates} \quad \Theta \quad \text{4 vector coordinates} \quad \Theta' \]

Taking transpose, we have \( \tilde{X}^T = \tilde{X}'^T \Theta^T \)

\[ \tilde{X}^T G \cdot X = \tilde{X}'^T \Theta^T G \cdot \Theta X' \]

\[ \| \text{since must be invariant} \| \]

\[ \tilde{X}'^T G \cdot X' = \tilde{X}'^T \Theta^T G \cdot \Theta X' \]

Must hold for all \( X \neq X' \).

Hence \( G = \Theta^T G \cdot \Theta \)

**Note:** This can be taken as a definition of Lorentz transforms ...

Like \( \Theta^T \Theta = I \), is a definition of 1 \( 3 \times 3 \) matrix, \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \), \( \Theta \)

\( 3 \times 3 \) matrix \( \Theta \), s.t. \( \Theta^T \Theta = I \); \( \det \Theta \Theta = 1 \Rightarrow (\det \Theta)^2 = 1 \),

since \( \det \Theta = \det \Theta \Rightarrow \det \Theta = \pm 1 \Rightarrow SO(3), \text{or if } +1 : \text{SO}(3) \).
**DEF:** The Lorentz group of transformations is defined as the set of all $4 \times 4$ real matrices $\hat{O}$ such that

$$\hat{O}^T G \hat{O} = G$$

where $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the Minkowski space metric.

Similar $\det \hat{O}^T G \hat{O} = \det G = -1$ implies $\det \hat{O}^T \det G \det \hat{O} = -1$

$$\Rightarrow \frac{1}{(\det \hat{O})^2} = +1$$

$$\det \hat{O} = \pm 1$$

If $\pm 1$ determinant allowed $O(1,3)$ if only $+1 \Rightarrow SO(1,3)$

For $SO(3)$:

$$G^{\text{rot}} = (1,1,1)$$

orthonormal

Thus, element of $G = (1,1,1,-1,-1)$
So: Einstein relativity =

\[ \text{Laws of physics invariant under } SO(1,3) \times T_4 \text{ symmetry transformations.} \]

Lorentz (space-time) translation invariance (ind. space + time)

(together, these form "Poincare group")

[us. nonrelativistic: \( SO(3) \times T_4 \times "Gallileo" \)]

very concise & precise statement of relativity principle.

We saw space-time coordinates transform as 4-vectors \( \{x^i\} = (c^t, \vec{\mathbf{r}}) \)

\[ \text{& inner product } x^i g_{ij} x^j \] (Einsteinian summation implied) is invariant under \( SO(1,3) \).

This \( x^i g_{ij} x^j \) is very cumbersome.

Since we'll use this product so often - we'll see that we'll have to require that the Lagrangians...

describing relativistic particles, E & M ---
must all be written I & O. Locally
invariant, just like the Newtonian
ones must be SO(3) must (i.e. all
LO products \( \vec{F}_i \cdot \vec{F}_j \) etc. etc.)
leave absorb the \( g_{ij} \) into either
\( \tilde{x}_i \) or \( x_i \):
\[ \tilde{x}_i g_{ij} x_j = \tilde{x}_j x_j = \tilde{x}_i x_i \]
\[ \tilde{x}_j = \tilde{x}_i g_{ij} = \tilde{x}_i g_{ji} \quad (g_{ij} = g_{ji}) \]
and
\[ x_i = g_{ij} x_j = g_{ji} x_j \]

\[ \tilde{x}_j = g_{ji} \tilde{x}_i \rightarrow \]
\[ \tilde{x}_0 = g_{00} \tilde{x}_0 + g_{01} \tilde{x}_1 + g_{02} \tilde{x}_2 + g_{03} \tilde{x}_3 = \tilde{x}_0 \]
\[ \tilde{x}_1 = g_{11} \tilde{x}_1 = - \tilde{x}_1 \]
\[ \tilde{x}_2 = g_{22} \tilde{x}_2 = - \tilde{x}_2 \]
\[ \tilde{x}_3 = g_{33} \tilde{x}_3 = - \tilde{x}_3 \]
So
\[ \tilde{x}_j x_j = \tilde{x}_0 \tilde{x}_0 + \tilde{x}_1 \tilde{x}_1 + \tilde{x}_2 \tilde{x}_2 + \tilde{x}_3 \tilde{x}_3 \]
\[ = \tilde{x}_0 \tilde{x}_0 - \tilde{x}_1 \tilde{x}_1 - \tilde{x}_2 \tilde{x}_2 - \tilde{x}_3 \tilde{x}_3 \]
\[ = c^2 \tilde{t} \tilde{t} - \vec{\tilde{r}} \cdot \vec{\tilde{r}} \]
So we have 4 vectors, e.g. 
\[ X = \{ x^i \} = (ct, \vec{r}) \], \quad X' = \hat{O} X 

But the notion is more general — just like 3 vectors: \( \vec{r} \) but also \( \vec{A} \) or \( \vec{E} \) 

Just like all 3 vectors transform as \( \vec{r} \) under \( \mathfrak{so}(3) \), all 4 vectors transform as \( \{ x^i \} \) (or \( X \)) under \( \mathfrak{so}(1,3) \).

The generalization of the \( (\vec{r}_1 \cdot \vec{r}_2) \) "dot" product to \( \mathfrak{so}(1,3) \) is
\[ \bar{X}^T X = \bar{x}_i \bar{x}^i = \bar{x}^i \bar{x}^j = \bar{x}^i g_{ij} \bar{x}^j \]
and is a "Lorentz scalar". Its value is the same in all frames.

As opposed to the "Lorentz vector" \( \{ x^i \} \) whose components are different in different frames.

There also exist "Lorentz tensors" — and you already saw one of them —
The "metric tensor"

\[ g_{ij} \] = \begin{pmatrix} 1 & \gamma_1 \\ \gamma_1 & 1 \end{pmatrix} \\
which has the property

\[ g^T g = g \]

so is invariant under lorentz transformations.

A rank-k tensor is an object that has k lorentz polices. Under lorentz transformations, every "index gets transformed as a vector" — just like \( g \) above.

6 simple examples

3-tensors: \( I_{ij} = \sum_{a=1}^{N} m_a \left( \delta_{ij} \hat{r}_a \cdot \hat{r}_a - \hat{r}_a \cdot \hat{r}_b \right) \) or \( I_{ij} = \int d^3\mathbf{r} \rho(\mathbf{r}) \left( \delta_{ij} \hat{r}^2 - \hat{r}_i \hat{r}_j \right) \) for continuous body of mass density \( \rho(\mathbf{r}) \).

Point is, kinetic energy of body w.r.t. \( \mathbf{T} = \sum_{ij} \delta_{ij} \mathbf{E}_{ij} \) is invariant under rotations of coordinate system \( T = \mathbf{R}^T \cdot \mathbf{I} \cdot \mathbf{R} \) in matrix notation.