kinematic effects on unrect. frame.

* elevator stops

* bus starts

F: rope acts on you
\[ \text{in} \ a = F \]

but it is not frame

\[ \vec{F}_f' \]
not done w/ rigid body quite yet —

— but first — noninertial frame

\[ \overline{R}(t) = \overline{V}(t) \]

most general case is shown in 25.4.25:

\[ (y'' y'' y'') \text{ some origin of } (x'y'z') \]

but rotates around it

\[ \text{also prescribed} \]

* e.g. if Sun is taken inertial \((XYZ)\), \((X'Y'Z')\) — Earth moving round Sun, \((X''Y''Z'')\) — a system rotating w/ the Earth

* going to an arbitrary frame is a change of coordinates, so action & E-L. eqns are obtained by changing coordinates in \( L \), by learning relation of \( x''y''z'' \) to \( XYZ \) once (dep. \( \overline{R}(t), \overline{V}(t) \))
original \[ L = \frac{1}{2} m \vec{v}^2 - U(\vec{r}) \quad \text{in} \quad (x'y'z') \]

then we have \[ \vec{r}' = \vec{r} - \vec{R}(t) \quad \text{position} \]
\[ \vec{v}' = \vec{v} - \vec{V}(t) \quad \text{velocity} \]

\[ L' = L(\vec{r}', \vec{v'}, \vec{V}(t)) \]

so \[ L' = \frac{1}{2} m (\vec{v'}^2 + \vec{V}(t)^2) - U(\vec{r}' + \vec{R}(t)) \]

\[ \frac{1}{2} m \vec{v'}^2 + m \vec{v'} \cdot \vec{V}(t) + \frac{1}{2} m \vec{V}(t)^2 \]

[\text{this is a total derivative} - \vec{V}(t) = \int_{t_0}^t \vec{a}(t) \, dt \quad \text{drop}]

\[ m \vec{v'} \cdot \vec{V}(t) = m \frac{d}{dt} (\vec{r}' \cdot \vec{V}(t)) - \vec{r}' \cdot m \frac{d}{dt} \vec{V}(t) \]

[\text{drop}]

\[ L' = \frac{1}{2} m \vec{v'}^2 - U(\vec{r}' + \vec{R}(t)) - m \vec{r}' \cdot \frac{d}{dt} \vec{V}(t) \]

- for a single particle
- for many, w/ \[ U(\vec{r}_i - \vec{r}_j) \]

would have \[ U(\vec{r}'_i - \vec{r}'_j) \] i.e. not explicitly \( t \) dependent

after step 1:

\[ \text{What are the EOM?} \quad m \frac{d\vec{v'}}{dt} = - \frac{\partial U(\vec{r} + \vec{R}(t))}{\partial \vec{r}} - m \frac{d\vec{V}(t)}{dt} \]

\( \Rightarrow \) in an accelerated frame \( \Rightarrow \) like a uniform potential field force applied to mass and opposite to acceleration
Now to step 2: for this step it is best to imagine that:

(a) there's more than one particle & (two-body) $U$ depends on $\mathbf{r}_i - \mathbf{r}_j = \mathbf{r}'_i - \mathbf{r}'_j$

(b) $U$ is independent of orientation of $X''Y''Z''$ wrt $X'Y'Z'$ (e.g. it is central w/ center @ $\bar{\mathbf{r}}(1)$)

A more general case can of course be handled, but leads to messy formulae; if only when needed!

(to find $\mathbf{F}$)

Now we need velocity of particle wrt $X''Y''Z''$, i.e. velocity relative to $X''Y''Z''$

Remember HW: we know for $\mathbf{P}$ arbitrary point w/ coordinates $(x_1, x_2, x_3)$ in $(X'Y'Z')$ & $(x'_1, x'_2, x'_3)$ in $(X''Y''Z'')$:

$$x''_i = \sum_{j=1}^{3} A_{ij} (\theta, \eta, \psi) x'_j$$ or $$x'_i = \sum_{j=1}^{3} A^{-1}_{ij} (\theta, \eta, \psi) x''_j$$

As opposed to rigid body, here $x''_i$ can depend on $x'_i$ as well.
So now \( \dot{x}' \) are the components of velocity w.r.t. \((x'y'z')\)-frame.

So we have

\[
\dot{x}' = \frac{d}{dt} x' = \sum_{\ell=1}^{3} \frac{d}{dt} \left( A^{-1}_{j\ell} (\theta, \varphi, \psi) \dot{x}_\ell'' \right)
\]

\[
\Rightarrow \sum_{\ell=1}^{3} \frac{d}{dt} \left( A^{-1}_{j\ell} (\theta, \varphi, \psi) \right) \cdot \dot{x}_\ell''
\]

This is the same as

in #2 of HW 5, as \( x'' \)

is not differentiated — you showed this

term was \( \dot{\mathbf{x}}' = \mathbf{x}''' \)

This is a new term: \( \dot{x}_e'' \) is

"components of velocity w.r.t. \( x''y''z'' \) in \( x'y'z' \)

\( A' \). \( \dot{x}'' \) is "components of same velocity w.r.t \( x'y'z' \)

So in vector notation,

we have

\[
\mathbf{v}' = \mathbf{v}''' + \mathbf{v}''
\]

velocity of

particle in \( x'y'z' \)

due to

rotation of \( x''y''z'' \) w.r.t. \( x'y'z' \)

(written even if @ rest in \( x''y''z'' \))
to do it, also note that

\[ \frac{\partial}{\partial t} \left( \mathbf{V} \times \mathbf{F} \right) = \mathbf{V} \times \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{F} \]

\( \frac{\partial}{\partial t} \left( \mathbf{V} \times \mathbf{F} \right) = \mathbf{V} \times \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{F} \)]

Focus on (3), every use of

\[ \mathbf{V} \times \mathbf{F} \]

are extra terms in \( L'' \) in the final

\[ L'' = \frac{1}{2} m \left( \frac{\partial \mathbf{V}}{\partial t} \right)^2 + m \mathbf{V} \times \mathbf{F} \]

\[ L'' = \frac{1}{2} m \left( \frac{\partial \mathbf{V}}{\partial t} \right)^2 + m \mathbf{V} \times \mathbf{F} \]

\[ \frac{\partial}{\partial t} \left( \mathbf{V} \times \mathbf{F} \right) = \mathbf{V} \times \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{F} \]

\[ \frac{\partial}{\partial t} \left( \mathbf{V} \times \mathbf{F} \right) = \mathbf{V} \times \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{F} \]

\[ \frac{\partial}{\partial t} \left( \mathbf{V} \times \mathbf{F} \right) = \mathbf{V} \times \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{F} \]

\[ \frac{\partial}{\partial t} \left( \mathbf{V} \times \mathbf{F} \right) = \mathbf{V} \times \left( \frac{\partial \mathbf{F}}{\partial t} \right) + \frac{\partial \mathbf{V}}{\partial t} \times \mathbf{F} \]
Quick proof of latter:

\[
\bar{\tau}''(\bar{r} \times \bar{r}'') = \sum_{ij} \epsilon_{ijk} R_j \tau_k''
\]

\[
= \sum_{ij} \epsilon_{ijk} \nu_i \nu_j \tau_k'' = \bar{\tau}''(\bar{r} \times \bar{r})
\]

\[
\frac{dL}{d\bar{\tau}''} = m \bar{r}'' \times \bar{r}'' + m \bar{\tau}''(\bar{r}^2) - m \bar{s} \bar{b} (\bar{r} \cdot \bar{r}''')
\]

\[-m \frac{d\bar{r}''}{dt} - \frac{\partial U}{\partial \bar{\tau}''}
\]

\[
= m \bar{r}'' \times \bar{r}'' + m(\bar{r} \times \bar{r}'') \times \bar{r}''
\]

\[-m \frac{d\bar{r}''}{dt} - \frac{\partial U}{\partial \bar{\tau}''}
\]

So \[
\frac{dL}{dt} = \frac{dL}{d\bar{\tau}''} \frac{d\bar{\tau}''}{dt} \]

\[
m \bar{\tau}'' + m \bar{r}'' \times \bar{r}'' + m \bar{r}'' \times \bar{r}'' = \]

\[
m \bar{\tau}'' = -\frac{\partial U}{\partial \bar{\tau}''} - m \frac{d\bar{r}''}{dt} - m \bar{r}'' \times \bar{r}'' + 2m \bar{r}'' \times \bar{r}''
\]

Coulomb, \(1 \bar{\tau}''\)

\[
\text{centripetal} \quad \text{!!} \quad \bar{r}
\]

\[
\text{potential force}
\]

\[
\text{non-uniform rotation inertial force}
\]

at this point it's good to trap the "sin \bar{r}'' + \bar{r}''"
If there's no translational or rotational acceleration, $\dot{\mathbf{V}} = \ddot{\mathbf{V}} = 0$

$$L = \frac{1}{2} m \ddot{V}^2 + m \ddot{V} \cdot \left( \ddot{\mathbf{r}} \times \mathbf{V} \right) + \frac{1}{2} m (\ddot{\mathbf{r}} \times \ddot{\mathbf{r}})^2 - U$$

energy is actually conserved $\rightarrow \mathbf{p} = \frac{dL}{dt} = m(\ddot{\mathbf{r}} + \mathbf{V} \times \ddot{\mathbf{r}})$

$$E = \mathbf{p} \cdot \mathbf{V} - L = m \ddot{V} \cdot \mathbf{V} + m (\ddot{\mathbf{r}} \times \mathbf{V}) \cdot \mathbf{V} - \frac{1}{2} m \ddot{V}^2 - m \ddot{V} \cdot (\ddot{\mathbf{r}} \times \ddot{\mathbf{r}}) - \frac{1}{2} m (\ddot{\mathbf{r}} \times \ddot{\mathbf{r}})^2 + U$$

$$E = \frac{1}{2} m \ddot{V}^2 + U - \frac{1}{2} m (\ddot{\mathbf{r}} \times \ddot{\mathbf{r}})^2$$

(centrifugal potential energy)