On to Ch. VI - Rigid body motion

What's a rigid body?

\[ m_1 \quad m_2 \quad m_3 \]

Total mass: \( m_1 + m_2 + m_3 \)

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Rigid rods, massless

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Or many more particles

---

Which don't move relative to each other.

In the case where the dots become a continuum, replace by

\[ \rho(\vec{r}) \, d^3 r \]

 ((* this is a density function, \( \rho \) is the mass per unit volume, \( \vec{r} \) is a position vector, \( d^3 r \) is an infinitesimal volume. ))

\[ = \# \text{ of particles} \quad \text{darker - denser} \]

\[ \text{in volume} \quad \int \int \int \]

\[ \text{around} \quad \vec{r} \]

Total mass: \( \int \int \int \rho(\vec{r}) \, d^3 r = M \)
How can we describe the position of a rigid body?

1. Some arbitrary point in the body, its coordinate describes the position of the body as a whole — could take the C.M., but any other point is ok.
   - This gives 3 degrees of freedom (i.e. coordinates).

2. But then, the body can be oriented in an arbitrary way, for example
   - \((x', y', z')\) a coordinate system moving with the body
     - (the masses \(1, 2, 3\) have fixed positions in \((x', y', z')\) system)
     - To describe the orientation of \((x'y'z')\) wrt \((x'y'z)\) system we require 3 other numbers — in general \((x'y'z')\) is rotated wrt \((x'y'z)\) and a rotation depends on 3 #s.
so, we need 6 coordinates to describe the motion of the rigid body.

Let us 1st imagine that \( \bar{R} \) is the C.M. position.

\[
\bar{R} = \frac{\sum m_i \bar{r}_i}{\sum m_i}, \quad m_i - \text{the masses comprising the body,} \\
\bar{r}_i - \text{their positions in (x, y, z) system}
\]

Let an arbitrary point of the body have radius vector \( \bar{p} \) in \((x, y, z)\) system and \( \bar{p}' \) in \((x', y', z')\) system.

\[ \text{The motion of this arbitrary point consists of:} \]

(a) a displacement of C.M. vector \( \bar{R} \) (d\(\bar{R}\))

(b) a rotation of \( \bar{p} \) around some direction \( \vec{n} \) (axis of rotation) by source angle \( \Delta \phi \) (d\(\phi \) = \( \vec{n} \cdot \text{d} \bar{p} \))

(b) corresponds to a rotation of \((x', y', z')\) system w.r.t \((x, y, z)\) system, i.e. a rotation of the entire body around its center of mass.
Thus we have

\[
\frac{d\vec{p}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{p}}{dt} \times \vec{r} = \vec{V} + \vec{\Omega} \times \vec{r}
\]

Remember when we studied rotational invariance and angular momentum, we argued

\[
\frac{d\vec{r}}{dt} = \frac{d\vec{\phi}}{dt} \times \vec{r}
\]

in the plane of \( d\vec{\phi} \)

\[
d\vec{\phi} \times \vec{r} = \vec{n} \times \vec{r} d\phi
\]

so dividing by \( dt \)

\[
\frac{d\vec{p}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{p}}{dt} \times \vec{r} = \text{velocity vector of } \vec{p} \text{ wrt } (x,y,z) \text{ frame (rigid - doesn't change)}
\]

\[
\text{velocity of c.m. velocity wrt c.m. in } (x,y,z) \vec{V} \vec{\Omega}
\]

\[
\text{frame } \vec{V}
\]

\[
\text{hence } \vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}
\]
\[ \mathbf{r} : \text{fixed in } (x', y') \quad \mathbf{r} = (A, 0, 0) \text{ here, a point } \text{ a distance } A \text{ away from origin.} \]

but in \((x, y)\): \[ \mathbf{\dot{r}} = \mathbf{\frac{d}{dt} r} = (A \cos \omega t, A \sin \omega t, 0) \quad (\mathbf{\mathbf{R}_{CM} = \varnothing}) \]

\[ \mathbf{v} = \mathbf{\frac{d}{dt} r} = \omega (-A \sin \omega t, A \cos \omega t, 0) \]

\[ = \omega (0, 0, 1) \times (A \cos \omega t, A \sin \omega t, 0) \]

\[ = \mathbf{\Sigma} \times \mathbf{r} = \mathbf{r} \]

\[ \text{since } (\mathbf{\mathbf{r} \times r}) = \mathbf{\sum} r_{z} - \mathbf{\sum} r_{y} \]

\[ -\omega A \sin \omega t \]

\[ e + r \]
where $\vec{V}$ is the velocity of CM and $\vec{R}$ is the angular velocity of the rotation of the body (here: w.r.t c.m.).

Now suppose origin is not a c.m., but instead at some point away from c.m.:

we have $\vec{p} = \vec{R} + \vec{r}$
as well as $\vec{p} = \vec{R} + \vec{\alpha}$

now we know

$\vec{dp} = d\vec{R} + d\vec{\omega} \times \vec{R}$ as already shown

and $\vec{r} = \vec{r} - \vec{\alpha}$ from picture, hence

$\vec{dp} = d\vec{R} + d\vec{\omega} \times \vec{R} - d\vec{\omega} \times \vec{\alpha}$
or $\vec{dp} = (d\vec{R} - d\vec{\omega} \times \vec{\alpha}) + d\vec{\omega} \times \vec{\alpha}$

(divide by $dt$)

$\vec{\nu} = \vec{V} - \vec{\omega} \times \vec{\alpha} + \vec{r} \times \vec{\omega} \times \vec{\alpha}$ (*
Now from \( \vec{\nu} = \vec{R} + \vec{\nu} \), we have

\[
\begin{align*}
\vec{\nu} &= \vec{V} + \vec{r} \times \vec{\omega} \\
\vec{\omega} &= \vec{\omega}\text{, angular velocity of the body in system } (x'y'z') \text{ w/ origin at } \vec{R}
\end{align*}
\]

where \( \vec{\omega} \) is now the angular velocity of the body in system \((x'y'z')\) w/ origin at \( \vec{R} \).

\[
\vec{V} + \vec{r} \times \vec{\omega} = \vec{V} - \vec{r} \times \vec{a}
\]

Compare \((*)\) w/ \((***)\)

\[
\begin{align*}
\vec{V} &= \vec{V} - \vec{r} \times \vec{a} \\
\vec{r} &= \vec{r}
\end{align*}
\]

**Moral:** If \( \vec{R} \) is not CM, then

1. Velocity of \( \vec{R} = \) CM velocity + \( \vec{r} \times (-\vec{a}) \)
2. Angular velocity of frame fixed "in the body" is, at any instant, independent of the chosen origin.

This velocity is called the "angular velocity of the body."
In our disk example of p. 109.1, this time, choose origin of body-fixed system away from C.M.

Now we have \( \ddot{\rho} = \ddot{\rho}' - \ddot{a} \)

\[ \ddot{r}' = (A-a)\cos\omega t, (A-a)\sin\omega t, 0 ) \]

\[ \ddot{a} = (-a\cos\omega t, -a\sin\omega t, 0) \]

\[ \frac{d}{dt} \frac{d\ddot{r}}{dt} = \frac{d\ddot{a}}{dt} + \frac{d\ddot{r}}{dt} \]

\[ = -\Omega \times \ddot{a} + \Omega \times \ddot{r} \]

\[ = \Omega \times (-\ddot{a}) + \Omega \times \ddot{r} \]

\[ \ddot{r} = \dddot{r} + \Delta \times \ddot{r} \]

\[ \downarrow \text{velocity of origin of } (x',y') \]

(\textit{NB: this is very general (well, we proved it), as at any moment every rotation is a rotation around an axis})
So, now that we can describe kinematics -
- position & velocity of an arbitrary point of the body w.r.t. a fixed coordinate system
- we need to do dynamics = Lagrangian $\rightarrow$ E.O.M.

**Kinetic energy** $T = \sum \frac{1}{2} m_i v_i^2$ of all particles of the body

A particle $\overrightarrow{u} = \overrightarrow{v} + \overrightarrow{\omega} \times \overrightarrow{r}$

velocity

velocity of origin of $\mathbf{(x'y'z')}$

position of particle $\mathbf{r} = (x, y, z)

a given moment (is how fast orientation of $\mathbf{(x'y'z')}$ changes w.r.t. $\mathbf{(x'y'z')}$) - eg. \(\overrightarrow{\omega}\)

Imagine body made of individual discrete particles labeled by $a$

\[
\overrightarrow{r}_a = \overrightarrow{v} + \overrightarrow{\omega} \times \overrightarrow{r}_a
\]

These are some $\mathbf{v}$ particle

Let a $\mathbf{v}$ particle have mass $m_a$

\[
T_a = \frac{1}{2} m_a \overrightarrow{v}_a^2, \quad T = \sum \frac{1}{2} m_a \overrightarrow{v}_a^2
\]

use energy of each particle

\[
T = \sum \frac{1}{2} m_a \overrightarrow{v}_a^2
\]

\[
\text{Total energy of body}
\]
So, for \( T \) we have

\[
T = \frac{1}{2} \sum_a m_a \left( \vec{v}_a \right)^2 = \frac{1}{2} \sum_a m_a \left( \vec{V} + \vec{r}_a \times \vec{\omega} \right)^2 =
\]

\[
= \frac{1}{2} \sum_a m_a \left( \vec{V}^2 + \vec{V} \cdot (\vec{r}_a \times \vec{\omega}) + (\vec{r}_a \times \vec{\omega}) \cdot (\vec{r}_a \times \vec{\omega}) \right)
\]

\[
= \frac{1}{2} \vec{V}^2 \left( \sum_a m_a \right) + \frac{1}{2} \left( \sum_a m_a \vec{r}_a \right) \cdot \vec{V} \times \vec{r}_a 
\]

This is zero if \((x', y', z')\)

has its origin at C.M. —

we'll assume this from now on

\[
+ \frac{1}{2} \sum_a m_a \left( \vec{r}_a \times \vec{\omega} \right) \cdot \vec{r}_a = 0
\]

Now \( \sum_a m_a = \mu = \text{total mass of body} \)

then use \((\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})\)

so rewrite \((\vec{r}_a \times \vec{\omega}) \cdot \vec{r}_a = \vec{J}^2 - (\vec{r}_a \times \vec{\omega})^2\)

\[
\vec{J}^2 = \frac{1}{2} \mu \frac{\vec{V}^2}{cM} + \frac{1}{2} \sum_a m_a \left( \vec{J}_a^2 - (\vec{r}_a \times \vec{\omega})^2 \right) = 0
\]

last term in \((\_\_\_)\) \Rightarrow \vec{J}^2 = \frac{3}{2} \sum_i \vec{J}_{a_i}^2 \quad \vec{r}_a = \sum_{i=1}^3 \vec{r}_{a_i}

\[
\vec{r}_a = \{ r_{a_1}, r_{a_2}, r_{a_3} \} \quad \vec{r}_{a_i}^2 = \sum_{i=1}^3 \vec{r}_{a_i}^2
\]

\[
(\text{instead of using } x, y, z)
\]
\[
\left( \overrightarrow{b} \cdot \overrightarrow{r}_a \right)^2 = \left( \sum_{i=1}^{3} \overrightarrow{b}_i \cdot \overrightarrow{r}_{a_i} \right)^2 = \\
= \left( \sum_{i=1}^{3} \overrightarrow{b}_i \cdot \overrightarrow{r}_{a_i} \right) \left( \sum_{j=1}^{3} \overrightarrow{b}_j \cdot \overrightarrow{r}_{a_j} \right) = \\
\sum_{i,j=1}^{3} \overrightarrow{b}_i \cdot \overrightarrow{r}_{a_i} \overrightarrow{b}_j \cdot \overrightarrow{r}_{a_j}
\]

\[
\sum_{i=1}^{3} \overrightarrow{r}_{a_i} \cdot \sum_{j=1}^{3} \overrightarrow{r}_{a_j} = \sum_{i=1}^{3} \overrightarrow{r}_{a_i} \cdot \sum_{j=1}^{3} \overrightarrow{r}_{a_j}
\]

so \[
\overrightarrow{b} \cdot \overrightarrow{r}_a - \left( \overrightarrow{b} \cdot \overrightarrow{r}_a \right)^2 = \sum_{i,j=1}^{3} \delta_{ij} \overrightarrow{b}_i \cdot \overrightarrow{b}_j \left( \sum_{k=1}^{3} \overrightarrow{r}_{a_k}^2 - \overrightarrow{r}_{a_i} \cdot \overrightarrow{r}_{a_j} \right)
\]

finally back to \( T \in \mathbb{R}^3 \)

\[
T = \frac{1}{2} \mu \overrightarrow{V}_c m + \frac{1}{2} \sum_{a} \sum_{i,j=1}^{3} \overrightarrow{R}_i \cdot \overrightarrow{R}_j \left( \delta_{ij} \overrightarrow{r}_a^2 - \overrightarrow{r}_{a_i} \cdot \overrightarrow{r}_{a_j} \right) = \\
= \frac{1}{2} \mu \overrightarrow{V}_c m + \frac{1}{2} \sum_{a} \sum_{i,j=1}^{3} \overrightarrow{R}_i \cdot \overrightarrow{R}_j \left( \sum_{k=1}^{3} m_k \delta_{ij} \overrightarrow{r}_a^2 - \overrightarrow{r}_{a_i} \cdot \overrightarrow{r}_{a_j} \right)
\]

This is a geometric characteristic of the body: depends on masses and their distribution only.
\[ I_{ij} = \sum \limits_a m_a (\delta_{ij} \mathbf{r}_a^2 - \mathbf{R}_i \mathbf{R}_j) \]

\( \sum \) over all particles

\( \mathbf{r}_a \) is position vector of the body

\( \mathbf{R}_i, \mathbf{R}_j \) for us, it means a simple thing

\( \mathbf{r}_a \) is component \((i = x, y, z)\) of the coordinate of \( a \)-th particle in \((x', y', z', \text{system whose origin assumed to be C.M.})\)

\( \mathbf{R}_i \text{, vector = column} \):

\[
\begin{pmatrix}
  r_x \\
  r_y \\
  r_z
\end{pmatrix}
\]

\( I_{ij} \) is matrix \((3 \times 3, \text{here})\)

\( I_{ij} \) is 2nd rank tensor

2 indices = 2nd rank tensor

In our case \( I_{ij} = I_{ji} \)

\( \text{Obvious - e.g. } I_{12} = \frac{2}{a} \sum \limits_a m_a \mathbf{r}_a \mathbf{r}_{a2} \)

\[ = - \sum \limits_a m_a \mathbf{r}_a \mathbf{r}_{a2} \mathbf{r}_{a2} = I_{21}, \text{ etc...} \]

- symmetric tensor of 2nd rank

(symmetric matrix, \(3 \times 3\))
Now, from linear algebra, you know that if you're given a symmetric \( n \times n \) matrix, it can always be diagonalized by an orthogonal \((n \times n)\) transformation.

- Here: \( n = 3 \)

\[
I_{\text{diag}} = O I O^T
\]

In matrix notation: \( 3 \times 3 \) symmetric matrix \( I_{ij} \) with \( m \times n \) elements \( I_{ij} \).

I.e. inertia tensor can be diagonalized by an orthogonal transformation.

The elements of \( I_{\text{diag}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \)

are called "principal moments of inertia" = eigenvalues of the \( 3 \times 3 \) symmetric \( \|I_{ij}\| = I \)

the orthogonal transformation \( O \) corresponds to a change of orientation of \((x', y', z')\) which ensures that
in the rotated system, \( I \) is diagonal.

The corresponding directions are called “principal axes of inertia” (i.e., \( x'y'z' \) of system where \( \mathcal{I} \) is diagonal).

Usually, it is evident from the symmetries of the body what the principal axes of inertia are; but for a body of an arbitrary shape one would have to rely on the math.

Here’s an example of an “unfortunate” choice of \( (x'y'z') \):

\[
\begin{align*}
\mathbf{R}_{(y')} & = \alpha \cos \Theta & \mathbf{R}_{(z')} & = \alpha \cos \Theta \\
\mathbf{R}_{(x')} & = -\alpha \sin \Theta & \mathbf{R}_{(z')} & = \alpha \sin \Theta \\
\mathbf{a}_{(z')} & = (0, 0, \alpha) & \mathbf{a}_{(z')} & = (0, 0, \alpha) \\
\end{align*}
\]

\[
I_{ij} = \sum_a m_a \left( \mathbf{r}_{a_i} \cdot \mathbf{r}_{a_j} - \mathbf{r}_{a_i} \cdot \mathbf{r}_{a_j} \right)
\]
so in this system we have

\[
I_{11} = m(a^2 - a^2 \cos^2 \Theta) + m(a^2 - a^2 \cos^2 \Theta)
\]

\[
= 2ma^2 \sin^2 \Theta
\]

\[
I_{12} = I_{21} = -m a^2 \cos \Theta \sin \Theta
\]

\[
I_{22} = m(a^2 - a^2 \sin^2 \Theta) + m(a^2 - a^2 \sin^2 \Theta)
\]

\[
= 2ma^2 \cos^2 \Theta
\]

\[
I_{13} = I_{31} = 0, \quad I_{23} = I_{32} = 0 \quad \text{(since } g_{1,2} = 0) \]

\[
I_{33} = 2ma^2
\]

So,

\[
I = 2ma^2 \begin{pmatrix}
\sin^2 \Theta & -\sin \Theta \cos \Theta & 0 \\
-\sin \Theta \cos \Theta & \cos^2 \Theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(A)

We wish to find the eigenvalues of \( \lambda \)

\[
\det(I - \lambda \mathbf{I}) = 0
\]

\[
\lambda(\sin^2 \Theta - \lambda)(\cos^2 \Theta - \lambda)
\]

\[
\sin \Theta \cos \Theta = \cos \Theta \sin \Theta = 0
\]

\[
\lambda^2 - \lambda(\sin^2 \Theta + \cos^2 \Theta) = 0
\]

\[
\lambda = 0, \quad \lambda = 1
\]

hence there exists a coordinate frame where

\[
I = 2ma^2 \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Physics - silly choice of orientation

\[ I_{33} = 2ma^2 \]

\[ I_{11} = 0 \]

\[ I_{22} = 2ma^2 \]

So generally we have

\[ T = \frac{1}{2} \mu V_{\text{cm}}^2 + \frac{1}{2} \sum_{i=1}^{3} \omega_i \cdot \mathbf{r}_i \cdot I_{ij} \]

Or in a \((x', y', z')\) where axes are aligned w/ the principal axes of inertia:

\[ T = \frac{1}{2} \mu V_{\text{cm}}^2 + \frac{1}{2} \left( \omega_1^2 I_1 + \omega_2^2 I_2 + \omega_3^2 I_3 \right) \]

Generally \( I_1, I_2, I_3 \) are different
- They are all the same for a sphere, for example (we call such a body a "spherical top")

- If two are equal - like for our ⊙☹, we have a "symmetrical top"

- Such bodies have cylindrical symmetry

- And the more general case is an "asymmetrical top"

Finally:

"Parallel axes theorem": \( I_{ij} \) in c.m. 

\[ I'_{ij} = \sum_a m_a (\delta_{ij} \overrightarrow{r}^2_a - \overrightarrow{r}_{a_i} \cdot \overrightarrow{r}_{a_j}) \]
\[
\sum m_a \left( \delta_{ij} \left( \mathbf{r}_a^2 - 2 \mathbf{r}_a \cdot \mathbf{b} + \mathbf{b}^2 \right) - \mathbf{r}_a \cdot \mathbf{r}_j - \mathbf{r}_a \cdot \mathbf{b}_j - \mathbf{r}_j \cdot \mathbf{b}_j \right) = \int \\
\text{vastly for some reason}
\sum m_a \mathbf{r}_a = 0
\]

\[
+ \sum m_a \left( \mathbf{b}^2 \delta_{ij} - \mathbf{b}_i \mathbf{b}_j \right) = I_{ij} + \mu \frac{1}{2} \delta_{ij} \left( \mathbf{b}^2 - \mathbf{b}_i \mathbf{b}_j \right) \\
= I_{ij}'
\]

\[
\text{(shifted: } \mathbf{r}_a = \mathbf{r}_a' + \mathbf{b} \text{)}
\]

→ What is \( L \) w/ external potential?

\[
L = T - U
\]

\[
T = \frac{1}{2} \mu \mathbf{v}_{cm}^2 + \frac{1}{2} \left( I_{x'} \mathbf{v}_{x'}^2 + I_{y'} \mathbf{v}_{y'}^2 + I_{z'} \mathbf{v}_{z'}^2 \right)
\]

\( U \) depends on \( \mathbf{R}_{cm} \) as well as orientation of \( (x'y'z') \) w.r.t. \((x'y'z')\)

e.g., representing force applied to body as a whole as well as see will represent forces on body.
For bodies w/ continuous mass density \( p(\vec{r}) \)

\[
\frac{p(\vec{r})}{\text{mass inside}} \, d\vec{r} = d^3\vec{r} \\
I_{ij} = \int p(\vec{r}) (\delta_{ij} \vec{r}^2 - \vec{r}_i \cdot \vec{r}_j) \, d^3\vec{r} \\
\text{volume of body}
\]

which is simply the continuum version of

\[
\sum_i m_i (\delta_{ij} \vec{r}_i^2 - \vec{r}_i \cdot \vec{r}_j)
\]

& \( \mu \) is given by same expression, w/ corresponding

\[
I_{ij} \text{ tor} \quad \mu = \int p(\vec{r}) \, d^3\vec{r}
\]

Our formulae so far can be used to calculate \( \mu \)

for various cases -- e.g.,

- **Homogeneous cylinder**: let's say moment of inertia \( I \) around
  principal axes || axes of cylinder and mass is \( \mu \).
  - let's say it's rolling on a plane without slipping.