A rigid cylinder of radius $R$ and mass $\mu$ has a moment of inertia $I$ around an axis going through the center of mass and parallel to the central axis of the cylinder. The cylinder is homogeneous along its central axis, but not in the radial and angular directions. Thus, its center of mass is displaced a distance $a$ ($0 < a < R$) from the central axis. The cylinder can roll without slipping on a plane and dissipation due to friction is assumed to be negligible. The plane is inclined on an angle $\theta$ with respect to the horizontal and the central axis of the cylinder is parallel to the horizontal plane.

1. Express the speed of the center of mass of the cylinder in terms of its angular velocity, angle of rotation, and the geometric parameters $a$, $R$, $I$. 

2. Find the kinetic energy of the cylinder.

3. Find the potential energy of the cylinder.

4. Find the Euler-Lagrange equation describing the motion of the cylinder.

5. Discuss qualitatively the motion of the cylinder for $\theta = 0$ and $\theta \neq 0$ and describe how the two cases differ.

Total marks for I.: 30 points
II. A particle moves in a circular orbit in the Kepler potential, \( U(r) = -\frac{k}{r} \). Show that if \( k \) suddenly decreases to half its original value, the particle’s orbit becomes parabolic.

*Total marks for II.: 20 points*

III. A particle of mass \( m \) is subject to a potential force \( U(|\vec{r}|) \). Consider a frame of reference uniformly rotating around \( \vec{r} = 0 \) with angular velocity \( \vec{\Omega} \). Find the Lagrangian and then the Hamiltonian of the particle in the rotating frame of reference. Explain the meaning of any \( \vec{\Omega} \)-dependent terms in the Hamiltonian you obtain.

*Total marks for III.: 20 points*

IV. Consider the scattering of a particle of mass \( m \) and energy \( E \) in an attractive potential well described by a potential \( U(\vec{r}) = -V_0 \) if \( |\vec{r}| < R \) and \( U(\vec{r}) = 0 \) if \( |\vec{r}| > R \).

1. Find the differential scattering cross section and sketch its angular dependence.

*22 points*

2. What is the total scattering cross section? Explain.

*8 points*

*Total marks for IV.: 30 points*

*Total marks for the exam 30 + 20 + 20 + 30 = 100*

*Total number of pages = 2*