Quantum entanglement and the phases of matter

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Sommerfeld-Bloch theory of metals, insulators, and superconductors: many-electron quantum states are adiabatically connected to independent electron states.
Sommerfeld-Bloch theory of metals, insulators, and superconductors: many-electron quantum states are adiabatically connected to independent electron states

**Band insulators**

An even number of electrons per unit cell
Sommerfeld-Bloch theory of metals, insulators, and superconductors: many-electron quantum states are adiabatically connected to independent electron states.

**Metals**

An odd number of electrons per unit cell
Sommerfeld-Bloch theory of metals, insulators, and superconductors:
many-electron quantum states are adiabatically connected to independent electron states.
Sommerfeld-Bloch theory of metals, insulators, and superconductors: many-electron quantum states are adiabatically connected to independent electron states
Modern phases of quantum matter

Not adiabatically connected to independent electron states:

*many-particle quantum entanglement*
Quantum superposition and entanglement
Quantum Entanglement: quantum superposition with more than one particle
Quantum Entanglement: quantum superposition with more than one particle

Hydrogen atom:
Quantum Entanglement: quantum superposition with more than one particle

Hydrogen atom:

Hydrogen molecule:

\[ = \frac{1}{\sqrt{2}} \left( |↑↓⟩ - |↓↑⟩ \right) \]

Superposition of two electron states leads to non-local correlations between spins
Quantum Entanglement: quantum superposition with more than one particle
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Quantum Entanglement: quantum superposition with more than one particle

Einstein-Podolsky-Rosen “paradox”: Non-local correlations between observations arbitrarily far apart
Quantum superposition and entanglement
Quantum superposition and entanglement

Quantum critical points of electrons in crystals

String theory and black holes
Quantum superposition and entanglement

Quantum critical points of electrons in crystals

String theory and black holes

Thursday, March 22, 2012
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \]

S=1/2 spins

Examine ground state as a function of \( \lambda \)
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

At large \( \lambda \) ground state is a “quantum paramagnet” with spins locked in valence bond singlets
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

Nearest-neighor spins are “entangled” with each other. Can be separated into an Einstein-Podolsky-Rosen (EPR) pair.
Spinning electrons localized on a square lattice

\[ H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j \]

For \( \lambda \approx 1 \), the ground state has antiferromagnetic ("Néel") order, and the spins align in a checkerboard pattern.
Spinning electrons localized on a square lattice

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For \( \lambda \approx 1 \), the ground state has antiferromagnetic ("Néel") order, and the spins align in a checkerboard pattern. No EPR pairs.
\[
\lambda = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)
\]
\[
\lambda_c = \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)
\]

A. Oosawa, K. Kakurai, T. Osakabe, M. Nakamura, M. Takeda, and H. Tanaka,
An insulator whose spin susceptibility vanishes exponentially as the temperature $T$ tends to zero.
TlCuCl$_3$

Quantum paramagnet at ambient pressure
TlCuCl$_3$

Neel order under pressure

\[ = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \]
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

"triplon"
Excitation spectrum in the paramagnetic phase

Spin $S = 1$

“triplon”
Excitation spectrum in the paramagnetic phase

Spin $S = 1$ 
“triplon”
Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase

Spin waves
Excitation spectrum in the Néel phase

Spin waves
Excitations of TlCuCl$_3$ with varying pressure

Excitations of TlCuCl$_3$ with varying pressure

Broken valence bond ("triplon") excitations of the quantum paramagnet

Excitations of TlCuCl$_3$ with varying pressure

Spin waves above the Néel state

Excitations of TlCuCl$_3$ with varying pressure

Higgs boson
First observation of the Higgs boson at the theoretically predicted energy!

S. Sachdev, arXiv:0901.4103

\[
\lambda = \frac{1}{\sqrt{2}} \left( |\uparrow \downarrow \rangle - |\downarrow \uparrow \rangle \right)
\]
Quantum critical point with non-local entanglement in spin wavefunction

Tensor network representation of entanglement at quantum critical point

D-dimensional space

The low energy excitations are described by a theory which has the same structure as Einstein's theory of special relativity, but with the spin-wave velocity playing the role of the velocity of light.

The theory of the critical point has an even larger symmetry corresponding to conformal transformations of spacetime: we refer to such a theory as a CFT3.

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- Long-range entanglement

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Quantum superposition and entanglement

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String theory and black holes
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String theory and black holes
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Quantum critical points of electrons in crystals

String theory and black holes
- Allows unification of the standard model of particle physics with gravity.

- Low-lying string modes correspond to gauge fields, gravitons, quarks ...
A $D$-brane is a $d$-dimensional surface on which strings can end.

- The low-energy theory on a $D$-brane has no gravity, similar to theories of entangled electrons of interest to us.
- In $d=2$, we obtain strongly-interacting CFTs. These are "dual" to string theory on anti-de Sitter space: AdS$_4$. 

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• In $d = 2$, we obtain strongly-interacting CFT3s. These are “dual” to string theory on anti-de Sitter space: AdS4.
Tensor network representation of entanglement at quantum critical point
String theory near a D-brane

d-dimensional space

Emergent direction of AdS4
Tensor network representation of entanglement at quantum critical point

Emergent direction of AdS4

Brian Swingle, arXiv:0905.1317
Measure strength of quantum entanglement of region A with region B.

\[ \rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \]

Entanglement entropy \( S_{EE} = -\text{Tr} (\rho_A \ln \rho_A) \)
Entanglement entropy

$d$-dimensional space

depth of entanglement
**Entanglement entropy**

Most links describe entanglement within A.

$d$-dimensional space

depth of entanglement
Entanglement entropy

$d$-dimensional space

Links overestimate entanglement between A and B

depth of entanglement
Entanglement entropy

$d$-dimensional space

Entanglement entropy = Number of links on optimal surface intersecting minimal number of links.

depth of entanglement
The entanglement entropy of a region $A$ on the boundary equals the minimal area of a surface in the higher-dimensional space whose boundary co-incides with that of $A$.

This can be seen both the string and tensor-network pictures

Brian Swingle, arXiv:0905.1317
$\text{AdS}_{d+2}$

$\mathbb{R}^{d,1}$

Minkowski

$CFT_{d+1}$

Quantum matter with long-range entanglement

Emergent holographic direction

J. McGreevy, arXiv0909.0518
Emergent holographic direction

$\text{AdS}_{d+2}$

$\mathbb{R}^{d,1}$

Minkowski

$\text{CFT}_{d+1}$
Quantum matter with long-range entanglement

$r$
AdS$_{d+2}$

Emergent holographic direction

Area measures entanglement entropy

CFT$_{d+1}$ Quantum matter with long-range entanglement

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Quantum critical point with non-local entanglement in spin wavefunction

\[ \lambda_c = \frac{1}{\sqrt{2}} \left( \left| \uparrow \downarrow \right> - \left| \downarrow \uparrow \right> \right) \]
Classical spin waves

Dilute triplon gas

Quantum critical


Thermally excited spin waves

Thermally excited triplon particles

Neel order

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Classical spin waves

Dilute triplon gas

Quantum critical


Thermally excited spin waves

Thermally excited triplon particles

Short-range entanglement

Neel order

\( T \)

\( \lambda \)

\( \lambda_c \)

Thermally excited spin waves

Thermally excited triplon particles

Neel order

Quantum critical
Classical spin waves

Dilute triplon gas

Quantum critical

Excitations of a ground state with long-range entanglement

Thermally excited spin waves

Thermally excited triplon particles

Neel order

Excitations of a ground state with long-range entanglement

Needed: Accurate theory of quantum critical spin dynamics

Thermally excited spin waves

Thermally excited triplon particles

Neel order
String theory at non-zero temperatures

A 2+1 dimensional system at its quantum critical point
String theory at non-zero temperatures

A 2+1 dimensional system at its quantum critical point

A “horizon”, similar to the surface of a black hole!
Objects so massive that light is gravitationally bound to them.
Black Holes

Objects so massive that light is gravitationally bound to them.

In Einstein’s theory, the region inside the black hole horizon is disconnected from the rest of the universe.

Horizon radius \( R = \frac{2GM}{c^2} \)
Around 1974, Bekenstein and Hawking showed that the application of the quantum theory across a black hole horizon led to many astonishing conclusions.
Quantum Entanglement across a black hole horizon
Quantum Entanglement across a black hole horizon
Quantum Entanglement across a black hole horizon

Black hole horizon

Black hole horizon
Quantum Entanglement across a black hole horizon
Quantum Entanglement across a black hole horizon

There is a non-local quantum entanglement between the inside and outside of a black hole.
Quantum Entanglement across a black hole horizon

There is a non-local quantum entanglement between the inside and outside of a black hole.
Quantum Entanglement across a black hole horizon

There is a non-local quantum entanglement between the inside and outside of a black hole.

This entanglement leads to a black hole temperature (the Hawking temperature) and a black hole entropy (the Bekenstein entropy).
A “horizon”, whose temperature and entropy equal those of the quantum critical point.

String theory at non-zero temperatures

A 2+1 dimensional system at its quantum critical point.
String theory at non-zero temperatures

A “horizon”, whose temperature and entropy equal those of the quantum critical point.

Friction of quantum criticality = waves falling into black brane.

A 2+1 dimensional system at its quantum critical point.
A 2+1 dimensional system at its quantum critical point

An (extended) Einstein-Maxwell provides successful description of dynamics of quantum critical points at non-zero temperatures (where no other methods apply)

A “horizon”, whose temperature and entropy equal those of the quantum critical point
Quantum superposition and entanglement

Quantum critical points of electrons in crystals

String theory and black holes
Metals, “strange metals”, and high temperature superconductors

Insights from gravitational “duals”
High temperature superconductors

$\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$
Iron pnictides:
a new class of high temperature superconductors
BaFe$_2$(As$_{1-x}$P$_x$)$_2$

Resistivity $\sim \rho_0 + AT^\alpha$


Superconductivity in \( \text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2 \)

Resistivity \( \sim \rho_0 + AT^\alpha \)

Short-range entanglement in state with Neel (AF) order


Superconductor

Bose condensate of pairs of electrons

Short-range entanglement

Resistivity
\[ \rho \sim \rho_0 + A T^\alpha \]

Superconductivity

BaFe\(_2\)(As\(_{1-x}\)P\(_x\))\(_2\)
Resistivity: $\rho \sim \rho_0 + AT^\alpha$

Ordinary metal (Fermi liquid)

Sommerfeld-Bloch theory of ordinary metals

Momenta with electron states occupied

Momenta with electron states empty
The electron density

Excitations near the Fermi surface are responsible for the familiar properties of ordinary metals, such as resistivity $\sim T^2$.
Resistivity \( \sim \rho_0 + AT^\alpha \)

\[
\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2
\]


**Resistivity**

\[
\rho \sim \rho_0 + A T^\alpha
\]

---


Classical spin waves

Dilute triplon gas

Quantum critical

Neel order

Ordinary Metal

$\lambda_c$
Classical spin waves

Dilute triplon gas

Quantum critical

Strange Metal

Ordinary Metal

Neel order

$\lambda_c$
Resistivity \( \sim \rho_0 + AT^\alpha \)

Strange Metal

\( \text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2 \)


Strange Metal

BaFe$_2$(As$_{1-x}$P$_x$)$_2$

Resistivity $\sim \rho_0 + AT^\alpha$


Excitations of a ground state with long-range entanglement

Resistivity \( \sim \rho_0 + AT^\alpha \)

Strange Metal

\( \text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2 \)


Key (difficult) problem:

Describe quantum critical points and phases of systems with Fermi surfaces leading to metals with novel types of long-range entanglement.
Challenge to string theory:

Describe quantum critical points and phases of metals
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Describe quantum critical points and phases of metals

Can we obtain gravitational theories of superconductors and ordinary Sommerfeld-Bloch metals?
Challenge to string theory:

Describe quantum critical points and phases of metals

Can we obtain gravitational theories of superconductors and ordinary Sommerfeld-Bloch metals?

Yes

S. Sachdev, Physical Review D 84, 066009 (2011)
Challenge to string theory:

Describe quantum critical points and phases of metals

Do the “holographic” gravitational theories also yield metals distinct from ordinary Sommerfeld-Bloch metals?
Challenge to string theory:

Describe quantum critical points and phases of metals

Do the “holographic” gravitational theories also yield metals distinct from ordinary Sommerfeld-Bloch metals?

Yes, lots of them, with many “strange” properties!
**Challenge to string theory:**

Describe quantum critical points and phases of metals

How do we discard artifacts, and choose the holographic theories applicable to condensed matter physics?
Challenge to string theory:

Describe quantum critical points and phases of metals

How do we discard artifacts, and choose the holographic theories applicable to condensed matter physics?

Choose the theories with the proper entropy density

Checks: these theories also have the proper entanglement entropy and Fermi surface size!

The simplest example of a “strange metal” is realized by fermions with a Fermi surface coupled to an Abelian or non-Abelian gauge field.
Fermi surface of an ordinary metal
Fermions coupled to a gauge field

- Area enclosed by the Fermi surface $\mathcal{A} = Q$, the fermion density

Fermions coupled to a gauge field

- Area enclosed by the Fermi surface $A = Q$, the fermion density

- Critical continuum of excitations near the Fermi surface with energy $\omega \sim |q|^z$, where $q = |k| - k_F$ is the distance from the Fermi surface and $z$ is the dynamic critical exponent.

Fermions coupled to a gauge field

- Area enclosed by the Fermi surface $A = Q$, the fermion density

- Critical continuum of excitations near the Fermi surface with energy $\omega \sim |q|^z$, where $q = |k| - k_F$ is the distance from the Fermi surface and $z$ is the dynamic critical exponent.

- The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T^{d_{\text{eff}}}/z$ with $d_{\text{eff}} = 1$.  

Holography of “strange metals”

J. McGreevy, arXiv0909.0518

Thursday, March 22, 2012
Consider the following (most) general metric for the holographic theory

\[
ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d}(z-1)/(d-\theta)} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)
\]

This metric transforms under rescaling as

\[
x_i \rightarrow \zeta x_i, \quad t \rightarrow \zeta, \quad ds \rightarrow \zeta^{\theta/d} ds.
\]

This identifies \(z\) as the dynamic critical exponent (\(z = 1\) for “relativistic” quantum critical points).

What is \(\theta\)? (\(\theta = 0\) for “relativistic” quantum critical points).
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\begin{align*}
x_i & \rightarrow \zeta x_i \\
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This metric transforms under rescaling as

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$$t \rightarrow \zeta^z t$$
$$ds \rightarrow \zeta^{\theta/d} ds.$$ 

This identifies $z$ as the dynamic critical exponent ($z = 1$ for “relativistic” quantum critical points).

What is $\theta$? ($\theta = 0$ for “relativistic” quantum critical points).
At $T > 0$, there is a “black-brane” at $r = r_h$.

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$.

The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$
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The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$.

Under rescaling $r \to \zeta^{(d-\theta)/d} r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z} = T^{d_{\text{eff}}/z}$$

where $\theta = d - d_{\text{eff}}$ measures “dimension deficit” in the phase space of low energy degrees of a freedom.
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Under rescaling $r \rightarrow \zeta^{(d-\theta)}/d_r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z} = T^{d_{\text{eff}}/z}$$

where $\theta = d - d_{\text{eff}}$ measures “dimension deficit” in the phase space of low energy degrees of a freedom.

For a strange metal should choose $\theta = d - 1$. 

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Holography of “strange metals”

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + \frac{r^{2\theta}/(d-\theta)}{dr^2 + dx_i^2} \right) \]

\[ \theta = d - 1 \]
Holography of “strange metals”

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- The entanglement entropy exhibits logarithmic violation of the area law, expected for systems with Fermi surfaces, only for this value of \( \theta \)!
Holography of “strange metals”

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- The co-efficient of the logarithmic term is consistent with the Fermi surface size expected from \( \mathcal{A} = Q \).

Holography of “strange metals”

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    ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + \frac{r^2\theta/(d-\theta)}{d-\theta} dr^2 + dx_i^2 \right)
\]

\[\theta = d - 1\]

- The entanglement entropy exhibits logarithmic violation of the area law, expected for systems with Fermi surfaces, only for this value of \(\theta\)!

- The co-efficient of the logarithmic term is consistent with the Fermi surface size expected from \(A = Q\).

- Many other features of the holographic theory are consistent with a boundary theory which has “hidden” Fermi surfaces of gauge-charged fermions.

Conclusions

Phases of matter with long-range quantum entanglement are prominent in numerous modern materials.
Simplest examples of long-range entanglement are at quantum-critical points of insulating antiferromagnets
Conclusions

More complex examples in metallic states are experimentally ubiquitous, but pose difficult strong-coupling problems to conventional methods of field theory.
Conclusions

String theory and gravity in emergent dimensions offer a remarkable new approach to describing states with long-range quantum entanglement.
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Much recent progress offers hope of a holographic description of “strange metals”